

Localization of 5D Elko Spinors on Minkowski Branes

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Abstract

Recently, a new spin-1/2 fermionic quantum field with mass dimension one—Elko was introduced as a candidate of dark matter. In this paper, we investigate the localization of the zero mode of 5D Elko on Minkowski branes by presenting the equations of the Elko zero mode. We consider the 5D Elko field in two cases: the 5D massless Elko field and the 5D Elko field with coupling term. For the 5D massless Elko field, the zero mode can be localized on Randall-Sundrum (RS) thin brane but it can not be localized on the majority of thick branes. In the second case, when we introduce the 5D mass term, there will exist bound Elko zero mode in RS model for the 5D mass M_{Elko} in the range $0 \leq M_{Elko}^2 < 2k^2$. And when we introduce the Yukawa coupling $\eta \bar{\lambda} \phi^2 \lambda$, the Elko zero mode can be successfully localized on some special thick branes for a particular coupling constant. It is different from the localization of the zero mode of the conventional Dirac spinor, where the coupling constant lies in a range.

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I. INTRODUCTION

The idea that our world is restricted in a 4D hyper-surface (brane) which is embedded in a multi-dimensional space-time (bulk) has drawn more and more attentions in recent years. Brane world theory can originate from the string/M theory. In the framework of brane scenarios, Standard-Model (SM) fields are bound to the brane, while the gravity propagates in the bulk. The possibility that the extra dimensions can not be compact [1–7] or be large [8–10] gives us a novel route to solve some long-standing problems in high-energy physics and cosmology, such as the hierarchy problem, i.e., the large difference between the electro-weak scale $M_{EW} \sim 1\text{TeV}$ and the Planck scale $M_{Pl} \sim 10^{16}\text{TeV}$ [9, 10], and the cosmological constant problem [2, 4, 11–15]. The famous Randall-Sundrum (RS) model was presented in the end of 90's [5, 6]. In the RS model, extra dimensions may not be compact any more, i.e., the size of extra dimensions can be even infinite. But the thickness of the ideal RS brane is zero. A more realistic brane should have thickness. A thick brane scenario is usually based on gravity coupled to a scalar field [16–21]. The scalar provides “materials” to generate the brane configuration. Thus the branes are generated naturally instead of by introducing delta functions by hand. More information about all kinds of thick brane solutions can be found in the review article [22].

An important and interesting issue in brane world scenarios is that how various bulk matter fields are localized on branes by a natural mechanism. What we do know is that the gravity [5, 6, 23] and massless scalar field [24] can be localized on branes of different types. While the spin 1 Abelian vector fields can be localized on some thick branes and 6D RS brane instead of 5D RS brane [25–28]. The localization of spin-1/2 fermion is very interesting. There can exist a single bound state and a continuous spectrum of massive KK modes with scalar-fermion coupling in some cases [29–37]. On some other thick branes, there can exist discrete KK modes (mass gaps) and continuous spectrum which starts at a positive mass [27, 28, 38–41]. In Refs. [42, 43], the spectra of 4D fermions on some symmetric and asymmetric thick branes and anti-de Sitter thick branes are constituted of bound KK modes. It was found that there exist fermion resonances on some thick branes, and the life-times of the resonances are decided by the structure of the branes, the coupling ways of fermion and background scalar field (for many forms of scalar in Yukawa coupling ways chosen as we want), and the coupling constant [39, 42, 44–49].

On the other hand, in 2005, Ahluwalia and Grumiller introduced a new quantum field which is a spin-1/2 fermionic quantum field with mass dimension one [50, 51]. It was named as Eigenspinoren des Ladungskonjugationsoperators (Elko) in German, i.e., eigenspinors of the charge conjugation operator. Elko belongs to non-standard Wigner classes [50, 52] and it will be better to understand Elko in the scope of Very Special Relativity (VSR) framework [53]. One of the consequences of mass dimension one is that Elko can interact with itself, gravity and Higgs doublet, but the mismatch of mass dimensions with Dirac fermions prevents it from entering the fermionic doublets of the SM [51]. In addition, Elko is a non-local field and the Lorentz symmetry is broken because there exists a preferred direction. Elko is local along this direction [54]. Elko can be used to investigate many cosmological problems such as ‘horizon problem’, the dark energy problem and so on. Ahluwalia and Grumiller suggested that Elko can be considered as a first-principle candidate of dark matter [50, 51]. All of these interesting properties of Elko have attracted more and more attentions [55–77]. Therefore, Elko is a new matter field we are interesting in. In the framework of brane scenarios, the localization of various matter fields except Elko on branes have been studied and the mass spectra also have been given. The peculiar properties of Elko may result in that its localization is very different from the ones of other matter fields. All of these researches motivate us to investigate the interesting problem that whether 5D Elko field can be localized on various kinds of branes. At the same time, among these brane models, the Minkowski (flat) one is the simplest brane model. For the first investigation about the localization of the new matter field, we choose the Minkowski branes as our subjects. Because of the difficulty of the separating variables, we just consider the localization of the zero mode of 5D Elko, i.e., the 4D massless Elko field. We will show that, Elko zero mode can be localized on RS model and on some special thick branes with coupling term.

The organization of the paper is as follows: In Sec. II, we briefly review the Elko quantum field. Then, in Sec. III, we discuss the localization of the zero mode of 5D massless Elko on various Minkowski branes by presenting the equation which the zero mode $\chi_0(z)$ satisfies. And in Sec. IV, we discuss the localization of the zero mode of 5D Elko on these Minkowski branes with Yukawa coupling. Finally, the conclusion is given in the last section.

II. REVIEW OF ELKO FIELD

Elko can not be expressed in Weinberg's formalism and it belongs to non-standard Wigner classes [50, 52]. But it can originate from VSR [53], which, by its defining features, is restricted to four subgroups of the Lorentz group. The four subgroups are $\mathfrak{t}(2)$, $\mathfrak{e}(2)$, $\mathfrak{hom}(2)$ and $\mathfrak{sim}(2)$ [53], where $\mathfrak{sim}(2)$ contains all generators among them. Elko in a representation space is the direct sum of two types of spinor representations with the respective generators given by:

$$\text{Tape a: } T_1^a \equiv K_x^a + J_y^a, \quad T_2^a \equiv K_y^a - J_x^a, \quad K_z^a, \quad \text{and} \quad J_z^a;$$

$$\text{Tape b: } T_1^b \equiv K_x^b + J_y^b, \quad T_2^b \equiv K_y^b - J_x^b, \quad K_z^b, \quad \text{and} \quad J_z^b.$$

Here $\vec{J}^a = \vec{J}^b = \vec{\sigma}/2$ and $\vec{K}^a = -\vec{K}^b = -i\vec{\sigma}/2$ are the generators of rotations and boosts, respectively.

In addition, Elko spinors obey the unusual property $(CPT)^2 = -\mathbb{I}$. Here charge conjugation C can be defined as

$$C = \begin{pmatrix} 0 & i\Theta \\ -i\Theta & 0 \end{pmatrix} K, \quad (1)$$

where K is the complex conjugation operator, and Θ is the spin one half Wigner time reversal operator satisfying $\Theta(\vec{\sigma}/2)\Theta^{-1} = -(\vec{\sigma}/2)^*$. Thus, the Θ can be given as

$$\Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2)$$

Elko spinors are eigenspinors of the charge conjugation operator: $C\lambda(k^\mu) = \pm\lambda(k^\mu)$ (k^μ is a polarization vector). From these properties, an Elko can be written as:

$$\lambda(k^\mu) = \begin{pmatrix} \pm i\Theta[\lambda^b(k^\mu)]^* \\ \lambda^b(k^\mu) \end{pmatrix}. \quad (3)$$

Here $\lambda^b(k^\mu)$ is a type-b Elko bispinor. It is clear that $\pm i\Theta[\lambda^b(k^\mu)]^*$ transforms as a type-a Elko bispinor. The plus (minus) sign generates the self conjugate spinors (anti-self conjugate spinors) which we denote by the symbol $\varsigma(k^\mu)$ ($\tau(k^\mu)$). Then, spanning this space by eigenspinors, we get

$$\lambda_+^b(k^\mu) = e^{i\vartheta_+} \sqrt{m} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (4)$$

$$\lambda_-^b(k^\mu) = e^{i\vartheta_-} \sqrt{m} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (5)$$

where the phases $\vartheta_{\pm} = \mp\phi/2$ with ϕ demanded by the quantum field [53]. Thus we get the four Elko spinors:

$$\varsigma_{\pm}(k^{\mu}) = \varsigma(k^{\mu})|_{\lambda^b \rightarrow \lambda_{\pm}^b}, \quad \tau_{\pm}(k^{\mu}) = \pm\tau(k^{\mu})|_{\lambda^b \rightarrow \lambda_{\mp}^b}. \quad (6)$$

It can always transform k^{μ} as p^{μ} (p^{μ} is a general vector and represents $(E, p \sin \theta \cos \phi, p \sin \theta \sin \phi, p \cos \theta)$) and $\lambda(p^{\mu})$ is also an Elko. The dual spinors for Elko are given as

$$\bar{\varsigma}_{\pm}(p^{\mu}) = \mp i[\varsigma_{\mp}(p^{\mu})]^{\dagger} \gamma^0, \quad \bar{\tau}_{\pm}(p^{\mu}) = \mp i[\tau_{\mp}(p^{\mu})]^{\dagger} \gamma^0. \quad (7)$$

When the Dirac operator $\gamma_{\mu}p^{\mu}$ acts on Elko spinors, the results are given as

$$\gamma_{\mu}p^{\mu}\varsigma_{\pm}(p^{\mu}) = \pm im\varsigma_{\mp}(p^{\mu}), \quad \gamma_{\mu}p^{\mu}\tau_{\pm}(p^{\mu}) = \mp im\tau_{\mp}(p^{\mu}). \quad (8)$$

As we know, it should be $\gamma_{\mu}p^{\mu}\psi_{\pm} \propto \psi_{\pm}$ if ψ is a Dirac spinor. Therefore, Elko spinors do not satisfy Dirac equation. On the other hand, via further discussion, it is found that Elko satisfies Klein-Gordon (KG) equation: $(p_{\mu}p^{\mu} - m^2)\lambda(p^{\mu}) = 0$. Thus, we get the Lagrangian density of Elko $\mathfrak{L}^{\text{free}} = \partial^{\mu} \bar{\lambda} \partial_{\mu} \lambda - m^2 \bar{\lambda} \lambda$ in 4D flat space-time. For a general curved space-time, the Lagrangian density should be written as [51, 55, 58]

$$\mathfrak{L}_{\text{Elko}} = \frac{1}{2} \left[\frac{1}{2} g^{\mu\nu} (\mathfrak{D}_{\mu} \bar{\lambda} \mathfrak{D}_{\nu} \lambda + \mathfrak{D}_{\nu} \bar{\lambda} \mathfrak{D}_{\mu} \lambda) \right] - V(\bar{\lambda} \lambda), \quad (9)$$

where $V(\bar{\lambda} \lambda)$ is the potential of Elko field including the mass term and \mathfrak{D}_{μ} represents covariant derivative.

III. LOCALIZATION OF 5D MASSLESS ELKO SPINORS ON MINKOWSKI BRANES

In this section, we will study the localization of free Elko spinors on Minkowski branes in 5D space-time. As is shown in last section, the Lagrangian density of Elko is similar to that of a scalar field. In fact, there are more formal resemblances as we will see below.

The metric describing 4D Minkowski branes embedded in a 5D bulk is generally assumed as

$$ds^2 = e^{2A(y)} \eta_{\mu\nu} dx^{\mu} dx^{\nu} + dy^2, \quad (10)$$

where $e^{2A(y)}$ is the warp factor and y the extra coordinate. Further, by performing the coordinate transformation

$$dz = e^{-A(y)} dy, \quad (11)$$

the metric (10) transforms to a conformally flat one

$$ds^2 = e^{2A}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2), \quad (12)$$

which is more convenient for discussing the localization of gravity and various matter fields.

First we start by considering the action of a massless Elko field λ coupled to gravity in 5D space-time,

$$S_{\text{Elko}} = \int d^5x \sqrt{-g} \mathfrak{L}_{\text{Elko}}, \quad (13)$$

here the Lagrangian density for the Elko field is recomposed as

$$\mathfrak{L}_{\text{Elko}} = \frac{1}{2} \left[\frac{1}{2} g^{MN} (\mathfrak{D}_M \bar{\lambda} \mathfrak{D}_N \lambda + \mathfrak{D}_N \bar{\lambda} \mathfrak{D}_M \lambda) \right]. \quad (14)$$

In this paper, $M, N \dots = 0, 1, 2, 3, 5$ and $\mu, \nu \dots = 0, 1, 2, 3$ denote the 5D and 4D spacetime indices, respectively; and $\bar{A}, \bar{B} \dots = 0, 1, 2, 3, 5$ and $a, b \dots = 0, 1, 2, 3$ denote the 5D and 4D local Lorentz indices, respectively. The covariant derivatives are

$$\mathfrak{D}_M \lambda = (\partial_M + \Omega_M) \lambda, \quad \mathfrak{D}_M \bar{\lambda} = \partial_M \bar{\lambda} - \bar{\lambda} \Omega_M, \quad (15)$$

where the tangent space connection Ω_M is defined as

$$\Omega_M = -\frac{i}{2} (e_{\bar{A}P} e_{\bar{B}}^N \Gamma_{MN}^P - e_{\bar{B}}^N \partial_M e_{\bar{A}N}) S^{\bar{A}\bar{B}}, \quad S^{\bar{A}\bar{B}} = \frac{i}{4} [\Gamma^{\bar{A}}, \Gamma^{\bar{B}}]. \quad (16)$$

Here $e_{\bar{M}}^{\bar{A}}$ is the vierbein and satisfies the orthonormality relations $g_{MN} = e_{\bar{M}}^{\bar{A}} e_{\bar{N}}^{\bar{B}} \eta_{\bar{A}\bar{B}}$, and the 5D Gamma matrixes $\Gamma^{\bar{A}}$ satisfy $\{\Gamma^{\bar{A}}, \Gamma^{\bar{B}}\} = 2\eta^{\bar{A}\bar{B}} \mathbb{I}$. In our paper the vierbein is given by

$$e_{\bar{M}}^{\bar{A}} = \begin{pmatrix} e^A \hat{e}^a_\mu & 0 \\ 0 & e^A \end{pmatrix}, \quad \hat{e}^a_\mu = \mathbb{I}. \quad (17)$$

Now we can get the non-vanishing components of the spin connection Ω_M for a flat brane:

$$\Omega_\mu = \frac{1}{2} \partial_z A \gamma_\mu \gamma_5. \quad (18)$$

As we have shown, the Lagrangian density for the Elko field is similar to the scalar field, thus the equation of motion for Elko is just like the scalar field as expected:

$$\frac{1}{\sqrt{-g}} \mathfrak{D}_M (\sqrt{-g} g^{MN} \mathfrak{D}_N \lambda) = 0. \quad (19)$$

By considering the conformally flat metric (12) and using the non-vanishing components of the spin connection (18), we can rewrite Eq. (19) as:

$$\begin{aligned} \frac{1}{\sqrt{-g}}\hat{\mathfrak{D}}_\mu(\sqrt{-g}\hat{g}^{\mu\nu}\hat{\mathfrak{D}}_\nu\lambda) + \left[\frac{1}{2}A' \left(\hat{\mathfrak{D}}_\mu(\hat{g}^{\mu\nu}\gamma_\nu\gamma_5\lambda) + \hat{g}^{\mu\nu}\gamma_\mu\gamma_5\hat{\mathfrak{D}}_\nu\lambda \right) \right. \\ \left. - \frac{1}{4}A'^2\hat{g}^{\mu\nu}\gamma_\mu\gamma_\nu\lambda + e^{-3A}\partial_z(e^{3A}\partial_z\lambda) \right] = 0. \end{aligned} \quad (20)$$

Here $\hat{g}_{\mu\nu}$ is the induced metric on the brane, and $\hat{\mathfrak{D}}_\mu\lambda = (\partial_\mu + \hat{\Omega}_\mu)\lambda$ with $\hat{\Omega}_\mu$ the spin connection constructed by the induced metric $\hat{g}_{\mu\nu}$. For our case of flat branes, $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$ and hence $\hat{\mathfrak{D}}_\mu = \partial_\mu$ and the equation of motion can be simplified to

$$\partial^\mu\partial_\mu\lambda - A'\gamma_5\gamma^\mu\partial_\mu\lambda - A'^2\lambda + e^{-3A}\partial_z(e^{3A}\partial_z\lambda) = 0. \quad (21)$$

Now we get the equation of motion of Elko for flat branes in 5D space-time. We have emphasized the similarity between the Elko field and the scalar field. Here, let us consider the equation of motion of a 5D massless scalar field. It is turned out for Minkowski branes to be

$$\partial^\mu\partial_\mu\Phi + e^{-3A}\partial_z(e^{3A}\partial_z\Phi) = 0, \quad (22)$$

from which one can investigate 4D scalar fields by the KK decomposition $\Phi = \sum_n \phi_n(x)h_n(z)e^{3A/2}$. Comparing (21) with (22), it is easy to find that there are two extra terms for Elko, i.e., $-A'\gamma_5\gamma^\mu\partial_\mu\lambda$ and $-A'^2\lambda$. The second term does not effect the procedure of separating variables with the KK decomposition

$$\lambda(x, z) = \sum_n \hat{\lambda}_n(x)\chi_n(z)e^{3A/2}. \quad (23)$$

In fact, it will result in the new characteristic of 4D Elko spinors. While the first one will be in conflict with separation of variables. We know that 4D Elko field satisfies Eq. (8) instead of the Dirac equation. Therefore we have

$$\gamma^\mu\partial_\mu\varsigma_\pm(x) = \pm m\varsigma_\mp(x), \quad \gamma^\mu\partial_\mu\tau_\pm(x) = \mp m\tau_\mp(x). \quad (24)$$

Here m is the 4D Elko mass. When we just consider the localization of 4D massless Elko, i.e, the zero mode of Elko, the term $-A'\gamma_5\gamma^\mu\partial_\mu\lambda$ will vanish. For this reason, in this paper we only consider the localization of the zero mode of 5D Elko. Thus the equation of motion (21) is rewritten as

$$\partial^\mu\partial_\mu\lambda_0 - A'^2\lambda_0 + e^{-3A}\partial_z(e^{3A}\partial_z\lambda_0) = 0. \quad (25)$$

Here $\lambda_0 = \hat{\lambda}_0(x)\chi_0(z)e^{3A/2}$. $\hat{\lambda}_0$ is 4D massless Elko and satisfies the equation of motion $\partial^\mu \partial_\mu \hat{\lambda}_0(x) = 0$, which is just the massless KG equation because the induced metric on the brane is the Minkowski one. Then we obtain the following equation for $\chi_0(z)$:

$$[-\partial_z^2 + V_{\lambda_0}(z)]\chi_0(z) = 0, \quad (26)$$

where the term V_{λ_0} is given by

$$V_{\lambda_0}(z) = \frac{3}{2}A'' + \frac{13}{4}A'^2. \quad (27)$$

In the conventional case of a 5D massless scalar field, the equation for the scalar KK modes h_n is a Schrödinger equation and the effective potential V_Φ is given by [26, 27]

$$V_\Phi(z) = \frac{3}{2}A'' + \frac{9}{4}A'^2. \quad (28)$$

The difference of the coefficients of the A'^2 between the V_{λ_0} and V_Φ will bring very interesting results about localization of the fields on branes. We suspect that the difference may obstruct the localization of Elko zero mode, and we will show the result by investigating some specific brane models.

For the Elko zero mode $\lambda_0 = \hat{\lambda}_0(x)\chi_0(z)e^{3A/2}$, $\bar{\lambda}_0 = \hat{\bar{\lambda}}_0(x)\chi_0(z)e^{3A/2}$, we have $\gamma^\mu \partial_\mu \lambda_0 = 0$ and $\partial^\mu \bar{\lambda}_0 \gamma_\mu = 0$. When Eq. (26) is satisfied and the following normalization condition is obeyed:

$$\int_{-\infty}^{\infty} dz \chi_0^2(z) = 1, \quad (29)$$

we may integrate over the extra dimension and reduce the full 5D action (13) to the standard 4D action of the massless Elko field:

$$S = \frac{1}{2} \int d^4x \partial_\mu \hat{\lambda}_0 \partial^\mu \hat{\lambda}_0. \quad (30)$$

Next we will consider various kinds of Minkowski brane solutions, and analyze the localization of the zero mode of 5D Elko on these branes by using Eqs. (26) and (29). As we know, there are thin and thick brane world models. we will investigate them in the following subsections respectively.

A. The thin brane

As the typification of thin brane models, we consider RS model, and investigate the localization of the zero mode of Elko on the RS brane.

In 1999, Randall and Sundrum introduced the famous RS model to solve the hierarchy problem [5, 6]. There are two types of RS model: the RSI and the RSII. The extra dimension is compact in RSI model so that the zero mode of Elko is indeed a bound mode in RSI case. So, we give our attention to the RSII model with a non-compact extra dimension.

The action in RSII model is [6]

$$\begin{aligned} S &= S_{gravity} + S_{brane}, \\ S_{gravity} &= \int d^4x \int dy \sqrt{-G} \left\{ -\Lambda + \frac{1}{2} R \right\}, \\ S_{brane} &= \int d^4x \sqrt{-g_{brane}} \{ V_{brane} + \mathfrak{L}_{brane} \}, \end{aligned} \quad (31)$$

where R is the 5D Ricci scalar, G_{MN} is the 5D metric, Λ and V_{brane} are cosmological terms in the bulk and boundary, respectively. G_{MN} is given by (10). The extra dimension y is non-compact and the solution is given by

$$A(y) = -k|y|, \quad (32)$$

where k is a positive real constant. This solution holds when the boundary and bulk cosmological terms are related by [6]

$$V_{brane} = 6k, \quad \Lambda = -6k^2. \quad (33)$$

Working with the conformal metric (12), the coordinate transformation (11) gives that $k|z| + 1 = e^{k|y|}$, and the V_{λ_0} (27) is given as

$$V_{\lambda_0} = \frac{19k^2}{4(1+k|z|)^2} - \frac{3k\delta(z)}{1+k|z|}. \quad (34)$$

The general solution of the equation (26) is given by:

$$\chi_0(z) = C_1(k|z| + 1)^{\frac{1}{2} + \sqrt{5}} + C_2(k|z| + 1)^{\frac{1}{2} - \sqrt{5}}. \quad (35)$$

where C_1, C_2 are integral parameters. In order to get localized Elko zero mode χ_0 on the thin brane, the normalization condition (29) should be satisfied, which indicates that $\chi_0(z)$ must be vanished when $z \rightarrow \pm\infty$. The first term of solution (35) will be divergent when

$z \rightarrow \pm\infty$, so C_1 should vanish. C_2 can be determined according to requirement of the normalization condition. Then we get the bound zero mode for 5D massless Elko field on the RSII brane:

$$\chi_0(z) = \sqrt{(-1 + \sqrt{5})k(k|z| + 1)^{\frac{1}{2} - \sqrt{5}}}. \quad (36)$$

Obviously in RSII brane model, the difference of the coefficients of the A'^2 between the V_{λ_0} and V_Φ will change the normalization constant and the exponent of polynomial but does not obstruct the localization of the Elko zero mode.

B. The thick brane

Next, we consider the localization of the zero mode of 5D massless Elko on Minkowski thick branes. As we know, there are various thick branes and their properties are also different with each other. We just consider the case that the V_{λ_0} (27) vanishes when $z \rightarrow \infty$. The majority of Minkowski thick brane solutions lead to this case, such as the solution with a single scalar field, non-minimally coupled scalar field, Weyl gravity model and so on [16–22, 78–81]. As examples, we just review two solutions: one is for a standard scalar field [18, 19, 22] and the other is for a scalar field non-minimally coupled to the Ricci scalar curvature [22, 78].

The thick brane action of a standard scalar coupled to gravity can be written as

$$S = \int d^5x \sqrt{-g} \left[\frac{1}{2}R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right]. \quad (37)$$

For the sine-Gordon potential

$$V(\phi) = \frac{3}{2}c^2[3b^2 \cos^2(b\phi) - 4 \sin^2(b\phi)], \quad (38)$$

and the Minkowski brane metric (10), the solution is given by [18, 19, 22]

$$e^A = \left[\frac{1}{\cosh(cb^2y)} \right]^{1/3b^2}, \quad \phi(y) = \frac{2}{b} \arctan \left[\tanh \left(\frac{3cb^2y}{2} \right) \right], \quad (39)$$

where b and c are parameters related to the brane thickness.

In addition, Refs. [22, 78] considered thick brane solutions of a scalar field non-minimally coupled to the Ricci scalar curvature, and the action is given by

$$S = \int d^5x \sqrt{-g} \left[f(\phi)R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right], \quad (40)$$

where $f(\phi)$ is a function of the scalar field ϕ . The above action is conformally related to the Einstein frame action with the Ricci scalar term $\frac{1}{2}R$ via the conformal transformation $g_{MN} \rightarrow 2\tilde{g}_{MN}f(\phi)$. With the coupling function

$$f(\phi) = \frac{1}{2}(1 - \xi\phi^2) \quad (41)$$

and the metric (10), for a non-zero coupling constant $\xi \neq 0$, the solution is given by [22, 78]

$$e^{A(y)} = (\cosh(ay))^{-\gamma}, \quad \phi(y) = \phi_0 \tanh(ay), \quad (42)$$

where the $\gamma = 2(\frac{1}{\xi} - 6)$, and $\phi_0 = a^{-1}\phi(0) = \sqrt{\frac{3(1-6\xi)}{\xi(1-2\xi)}}$. The parameter ξ satisfies $0 < \xi < 1/6$, which means that the $\gamma > 0$.

We write the two solutions for the warp factor in a unified form

$$e^{2A} = \cosh(\alpha y)^{-2\beta}, \quad (43)$$

where β is a positive real constant and α an arbitrary constant parameter. We use (43) to analyze the localization of the Elko zero mode on these thick branes. The warp factor $e^{2A(y)}$ is a function of the extra coordinate y . But Eq. (26) should be expressed with the conformally flat coordinate z . So, we need the relation between z and y , which are related by the coordinate transformation (11) and given by

$$z(y) = -i \frac{\sqrt{\pi} \Gamma(\frac{1+\beta}{2})}{2|\alpha| \Gamma(1 + \frac{\beta}{2})} + i \text{sign}(\alpha y) \frac{[\cosh(\alpha y)]^{1+\beta}}{\alpha(1+\beta)} F, \quad (44)$$

where F is the hypergeometric function

$$F = {}_2F_1 \left[\frac{1}{2}, \frac{1+\beta}{2}, \frac{3+\beta}{2}, \cosh^2(\alpha y) \right]. \quad (45)$$

Here we face the difficulty that for general α and β we can not get an analytical form of $y(z)$ from the function $z(y)$ given in (44). But we can write the V_{λ_0} as a function of y :

$$V_{\lambda_0}(z(y)) = e^{2A} \left(\frac{3}{2} \frac{\partial^2 A}{\partial y^2} + \frac{19}{4} \left(\frac{\partial A}{\partial y} \right)^2 \right). \quad (46)$$

As is shown in figure 1, we can find that $z(y)$ is a monotonic function. It means that $V_{\lambda_0}(z)$ has the similar shape and property to $V_{\lambda_0}(z(y))$.

Now we consider the massless mode $\chi_0(z)$, for which Eq. (26) can be written in the extra coordinate y as

$$[-e^{2A} \partial_y^2 - e^{2A} A' \partial_y + V_{\lambda_0}(z(y))] \chi_0(z(y)) = 0, \quad (47)$$

where the zero mode $\chi_0(z(y))$ will have the similar figure and property to $\chi_0(z)$. Let $\chi_0(z(y)) = e^{-\frac{1}{2}A(y)}\rho(y)$, the above equation is reduced to

$$[-\partial_y^2 + 5\alpha^2\beta^2 - \alpha^2\beta(2 + 5\beta)\text{sech}^2(\alpha y)]\rho(y) = 0. \quad (48)$$

The general solution is given by

$$\rho(y) = C_1 P_{q-1}^{\sqrt{5}\beta}(\tanh(\alpha y)) + C_2 Q_{q-1}^{\sqrt{5}\beta}(\tanh(\alpha y)), \quad (49)$$

where $q(q-1) = \beta(2+5\beta)$, P and Q are the first and second Legendre functions, respectively. Hence, we get the solution of the massless mode

$$\chi_0(y) = \cosh(\alpha y)^{\beta/2} \left[C_1 P_{q-1}^{\sqrt{5}\beta}(\tanh(\alpha y)) + C_2 Q_{q-1}^{\sqrt{5}\beta}(\tanh(\alpha y)) \right]. \quad (50)$$

For arbitrary $\beta > 0$, $\cosh(\alpha y)^{\beta/2}$ will be divergent when $y \rightarrow \pm\infty$. So the normalization condition (29) require that $\rho(y)$ should vanish when $y \rightarrow \pm\infty$ if we want get a bound state. From the solution (49), $\rho(y)$ is a summation of two Legendre functions. According to the theory of the special functions, we know that the Legendre functions $P_{q-1}^{\sqrt{5}\beta}(\tanh(\alpha y))$ and $Q_{q-1}^{\sqrt{5}\beta}(\tanh(\alpha y))$ are convergent only under some strong restrictions. For the first Legendre function P , it requires that $q-1$ and $\sqrt{5}\beta$ are integers, or $q - \sqrt{5}\beta$ or $q - 1 - \sqrt{5}\beta$ is zero or negative integer just while $\text{Re}(\sqrt{5}\beta) < 0$. For the second Legendre function Q , it requires that both $q-1$ and $\sqrt{5}\beta$ are positive half odd integers when $\text{Re}(\sqrt{5}\beta) > 0$, or $q - 1 - \sqrt{5}\beta$ is a negative integer but $\sqrt{5}\beta$ is not an integer while $\text{Re}(\sqrt{5}\beta) < 0$. Here we can solve the equation $q(q-1) = \beta(2+5\beta)$ and get $q = \frac{1}{2}(1 \pm \sqrt{1+8\beta+20\beta^2})$. Obviously, the solution can not converge at $y = \pm\infty$ since these strong restrictions can not be satisfied. Thus we can not get a bound Elko zero mode. So for 5D massless Elko field, the 4D massless Elko can not be localized on these Minkowski thick branes.

IV. LOCALIZATION OF 5D ELKO SPINORS WITH COUPLING TERM ON MINKOWSKI BRANES

In this section, we will study the localization of the zero mode of Elko spinors with coupling term on Minkowski branes in 5D space-time. As is shown in last section, for the free Elko field in 5D space-time, we can not get a bound Elko zero mode on Minkowski thick branes. As the localization of Dirac spinors on brane, we introduce the interaction of Elko

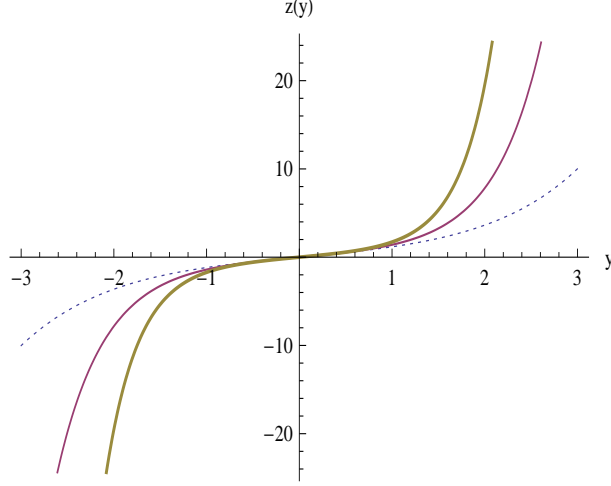


FIG. 1. The shapes of the function $z(y)$. The parameters are set to $\alpha = 1$, $\beta = 1$ for dashed line, $\beta = 2$ for thin line, and $\beta = 3$ for thick line.

spinors with the background scalar (the kink scalar), and the simplest choice is the Yukawa coupling. The Lagrangian density for the Elko field is recomposed as

$$\mathfrak{L}_{\text{Elko}} = \frac{1}{2} \left[\frac{1}{2} g^{MN} (\mathfrak{D}_M \bar{\lambda} \mathfrak{D}_N \lambda + \mathfrak{D}_N \bar{\lambda} \mathfrak{D}_M \lambda) \right] - \eta F(\phi) \bar{\lambda} \lambda, \quad (51)$$

where the $F(\phi)$ is a function of the background scalar field ϕ and η is the coupling constant. When $F(\phi)$ is a constant, the additional term in 51 is a mass term with $M_{\text{Elko}}^2 = \eta F(\phi)$. Then we can get the equation of motion for the Elko field coupled with the scalar:

$$\frac{1}{\sqrt{-g}} \mathfrak{D}_M (\sqrt{-g} g^{MN} \mathfrak{D}_N \lambda) - 2\eta F(\phi) \lambda = 0. \quad (52)$$

By considering the conformally flat metric (12), using the non-vanishing components of the spin connection for the flat branes (18) and just paying attention to the 4D massless Elko field (zero mode): $\gamma^\mu \partial_\mu \lambda = 0$, we can rewrite Eq. (52) as

$$\partial^\mu \partial_\mu \lambda - A'^2 \lambda + e^{-3A} \partial_z (e^{3A} \partial_z \lambda) - 2\eta e^{2A} F(\phi) \lambda = 0. \quad (53)$$

Then, by the KK decomposition $\lambda_0(x, z) = \hat{\lambda}_0(x) \chi_0(z) e^{3A/2}$, we obtain the equation for $\chi_0(z)$:

$$[-\partial_z^2 + V_{\lambda_0}(z)] \chi_0(z) = 0. \quad (54)$$

Here the V_{λ_0} is given by

$$V_{\lambda_0}(z) = \frac{3}{2} A'' + \frac{13}{4} A'^2 - 2\eta e^{2A} F(\phi). \quad (55)$$

As discussed in the last section, when Eq. (54) and normalization condition $\int_{-\infty}^{\infty} dz \chi_0^2(z) = 1$ are satisfied, the 5D action for the Elko zero mode (51) can be reduced to the standard 4D action of the massless Elko field (30).

As we emphasized, 5D free Elko fields can not be localized on Minkowski thick branes. We have known that the effective potential of Schrödinger equation for the 5D massless scalar KK modes is given by Eq. (28). The zero mode of 5D massless scalar can be localized because the corresponding Schrödinger equation can be factorized, which is the result of the fact that the coefficient of A'^2 is the square of the coefficient of A'' . Obviously, for Minkowski thick branes, the difference of the coefficients of the A'^2 between the V_{λ_0} and V_{Φ} would prevent the factorization of the V_{λ_0} and hence the localization of the Elko zero mode. When an appropriate $F(\phi)$ is introduced, the coefficient of the A'^2 may be adjusted to be the same as one of the scalar so that there exists the bound Elko zero mode. Then we can assume:

$$\begin{aligned} V_{\lambda_0}(z) &= \frac{3}{2}A'' + \frac{13}{4}A'^2 - 2\eta e^{2A}F(\phi) \\ &= \frac{3}{2}A'' + \frac{9}{4}A'^2, \end{aligned} \quad (56)$$

which is equivalent to

$$(\partial_z A(z))^2 - 2\eta e^{2A(z)}F(\phi) = 0. \quad (57)$$

It is more clear when the equation is written in extra coordinate y :

$$(\partial_y A(y))^2 = 2\eta F(\phi(y)). \quad (58)$$

This equation depends on the warp factor $e^{2A(y)}$, the scalar field ϕ and the function $F(\phi)$. It is reasonable to consider the scalar-Elko coupling $\eta \bar{\lambda} \phi^n \lambda$. Hence $F(\phi)$ can be taken as ϕ^n and $F(\phi(y))$ should be an even function of y according to Eq. (58). As we know, for majority of the brane world models, the scalar field ϕ is a kink, i.e., it is an odd function of y , so the simplest case is $n = 2$. Then we have:

$$A'^2(y) = 2\eta \phi^2(y), \quad (59)$$

or

$$A'(y) \propto \phi(y). \quad (60)$$

If the warp factor $e^{2A(y)}$ and the scalar field ϕ of a brane world model are related by Eq. (60), the Elko zero mode, i.e., the massless Elko particle may be localized on the brane world. It is exciting that there are many models can satisfy this equation. We will discuss them in following subsections respectively.

A. The thin brane

First, as the typification of thin brane models, we still consider the RSII model. RSII model is a thin brane world model, i.e, the thickness of the ideal RS brane model is zero. From the Eq. (31), it is clear that there not exist a scalar field ϕ to make brane. Here we introduce the 5D mass term, i.e, $\eta\phi^2 = M_{Elko}^2$ with M_{Elko} the 5D Elko mass. At the same time, notice that $A'(y) = -k \text{sign}(y)$ and $A'^2(y) = k^2$. It is natural that

$$M_{Elko}^2 \propto k^2. \quad (61)$$

For an arbitrary constant M_{Elko} , the V_{λ_0} (27) is read as:

$$V_{\lambda_0} = \frac{19k^2}{4(1+k|z|)^2} - \frac{2M_{Elko}^2}{(1+k|z|)^2} - \frac{3k\delta(z)}{1+k|z|} = \frac{(19-8\epsilon)k^2}{4(1+k|z|)^2} - \frac{3k\delta(z)}{1+k|z|}. \quad (62)$$

Here $\epsilon = M_{Elko}^2/k^2$. We can get the bound Elko zero mode by solving the equation (26):

$$\chi_0 = \sqrt{(-1 + \sqrt{5-2\epsilon})k(1+k|z|)^{\frac{1}{2}-\sqrt{5-2\epsilon}}}. \quad (63)$$

Here it is required that $0 \leq \epsilon < 2$. So, for any $0 \leq M_{Elko}^2 < 2k^2$, there can exist bound Elko zero mode on the RSII brane model.

B. The thick brane

Generally, for thick brane models based on the general relativity, Eq. (60) can not be satisfied. Fortunately it can be satisfied in some thick branes within modified gravity theory, for example, the thick brane solution of a scalar field non-minimally coupled to the Ricci scalar curvature [22, 78]. The solution is given as $e^{A(y)} = (\cosh(ay))^{-\gamma}$ and $\phi(y) = \phi_0 \tanh(ay)$. Obviously, the derivative of the wrap factor $A(y)$ and the scalar field are related by

$$A'(y) = -\gamma \partial_y \ln[\cosh(ay)] = -\alpha \gamma \tanh(ay) = -\frac{\alpha \gamma}{\phi_0} \phi(y). \quad (64)$$

Let $\eta = \frac{\alpha^2 \gamma^2}{2\phi_0^2}$ then Eq. (59) is satisfied.

When Eq. (59) is satisfied, the V_{λ_0} (55) can be rewritten as (56). After factorizing Eq. (26), we have

$$\left[\partial_z + \frac{3}{2} A'(z) \right] \left[-\partial_z + \frac{3}{2} A'(z) \right] \chi_0(z) = 0. \quad (65)$$

Then we get the bound Elko zero mode:

$$\chi_0(z) = C e^{\frac{3}{2} A(z)}, \quad (66)$$

where C is the normalization constant. For the RSII model, the zero mode is

$$\chi_0(z) = \sqrt{k} (k|z| + 1)^{-\frac{3}{2}}. \quad (67)$$

For the thick brane model [22, 78], it is

$$\chi_0(z(y)) = C \cosh(\alpha y)^{-\frac{3}{2}\gamma}. \quad (68)$$

The normalization constant C depends on the parameters γ and α . When $\gamma = 1$, $\alpha z = \sinh(\alpha y)$ and the zero mode is

$$\chi_0(z) = \frac{\alpha}{2} [1 + (\alpha z)^2]^{-\frac{3}{4}}. \quad (69)$$

It is clear that for the thick brane models which satisfy Eq. (60), only if it involves a particular coupling constant, there will exist bound Elko zero mode. The corresponding results are plotted in figure 2.

V. CONCLUSION AND DISCUSSION

In this paper, we have investigated the localization of Elko field, a spin-1/2 fermion field with mass dimension one, on flat branes in 5D space-time. First, we briefly review some fundamental structures of Elko, which shows that Elko is very different from a usual Dirac fermion field but it is similar to a scalar field on many aspects—the Lagrangian density and mass dimension, for example. Then, we consider various kinds of Minkowski branes and analyze the localization of the zero mode of Elko field on these branes by presenting equation of motion of the Elko zero mode. We consider the 5D Elko field in two cases: the 5D massless free Elko field and the 5D Elko field with coupling term. In the first case, we find that the Elko zero mode can be localized on the RSII brane but can not be localized on

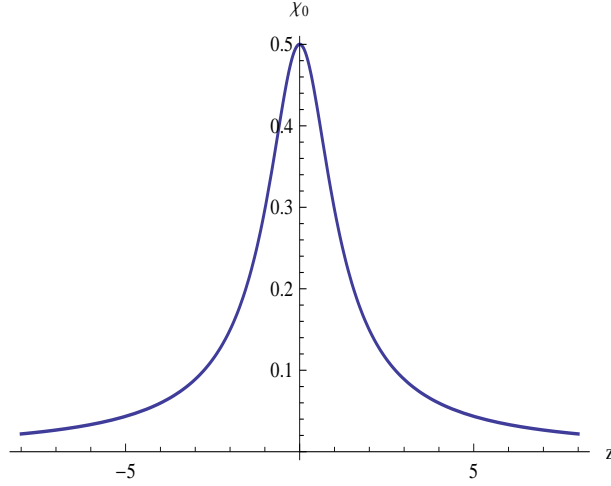


FIG. 2. The shape of the Elko zero mode $\chi_0(z)$ in the thick brane model with the scalar field non-minimally coupled to the Ricci scalar curvature. The parameters are set to $\gamma = 1$ and $\alpha = 1$.

the majority of Minkowski thick branes. In the seconde case, the Elko zero mode still can be localized on the RSII brane when the 5D Elko mass M_{Elko} is satisfied $0 \leq M_{Elko}^2 < 2k^2$, and it can also be localized on some specific thick branes which satisfy Eq. (60) when we introduce the Yukawa coupling $\eta \bar{\lambda} \phi^2 \lambda$. The coupling constant η should be a particular constant and this conclusion is very different from that of the conventional Dirac spinor, where the coupling constant lies in a range.

There are still two open issues. First, in order to investigate the localization of 4D massive Elko field, some new mechanisms need to be introduced to simplify Eq. (21) so that one can decompose the 4D part and the extra dimensional part. Second, for de Sitter branes and anti-de Sitter branes, there still exists the difficulty of separating variables from Eq. (19) or (20). We leave these problems in our further works.

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